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Field Assessment Problems

THE DIFFUSION OF POLYDISPERSE PARTICULATE CLOUDS

By

Ben Davidson
Leon Herbach

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FOR ERRATA

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THE FOLLOWING PAGES ARE CHANGES

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ERRATA

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The Diffusion of Polydisperse Particulate Clouds

Page 23 Title in Figure 5 should read "$\nu = 0.57$" instead of "$\nu = 0.53$.

Page 24 Legend in Figure 6 should read

— — ——— Neglecting Atmospheric Diffusion,

- - - - - - - - Analytic Approximation.

———————— Numerical Integration.

Note that this legend should be different from that in Figures 3 - 5.

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THE DIFFUSION OF POLYDISPERSE PARTICULATE CLOUDS

by

Ben Davidson
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GLOSSARY

		Equation
c	$= c(x, z)$, instantaneous concentration of diffusing particles at (x, z)	(1)
D	= deposition, $[gm/(m^2 sec)]$	(7)
D_m	= maximum deposition, $[gm/(m^2 sec)]$	(11), (21), (27)
D_o	= deposition for zero atmospheric diffusion	(19), (28)
f	$= H/\eta$ = dominant length scale	(8)
f^*	$= H/(\eta + \eta_o)$	(24), (38)
f_o	$= H/\eta_o$	(28)
F	= instantaneous flux of particles	(1)
h	= height of source	(6)
H	$= h\,\bar{u}(h)/(1 + \alpha)$	(8)
k	= von Karman's constant ≈ 0.4	(6)
k_1, k_o	= proportionality constants, function of ν only	(23), (25), (28), (34)
K	= diffusion coefficient $= u^* kz$: adiabatic case	(6)
	$= u^* k\epsilon z$: diabatic case	(53)
ℓ	= mixing length	(5), (22)
p	$= \Omega/\eta$	(8)
p^*	$= \Omega^*/(\eta + \eta_o)$	(24), (37)
p_o	$= \Omega_o/\eta_o$	(28)
Q	= source strength, $[gm/(m\ sec)]$	(7)
$Q(\Omega)$	= distribution of mass, as function of particle terminal velocity	(13)
R^*	= ratio of x_m predicted by (20), or (29) which neglects atmospheric diffusion to that predicted by (26), the analytic approximation.	(39)

		Equation
$\bar{R}^*[\tilde{R}^*]$	= ratio of x_m predicted by (9), which neglects dispersion of terminal velocities to location predicted by the analytic approximation.	(43), [(44)]
t	= time	(2)
$\bar{u}(z)$	= mean wind speed at height z	(4)
u^*	= friction velocity = $(\text{surface stress/density})^{1/2}$	(6)
$\bar{U}$	$= \frac{1}{h}\int_0^h \bar{u}(h)\,dh$	(55)
V	= instantaneous vector velocity of air	(1)
w	= vertical component of atmospheric velocity	(4)
x	= distance downwind	(4)
x_m	= point where maximum deposit occurs	(9), (20), (26)
z	= vertical distance	(6)
z_o	= roughness parameter	(6)
α	= parameter determined by z_o and depth of layer concerned	(6)
β	= stability parameter	(51)
$\Gamma(\ )$	= gamma function	(7)
Δ	= dimensionless deposition	(7)
Δ_m	= maximum deposition, [dimensionless]	(10), (21), (27)
Δ_o	= dimensionless deposition for zero atmospheric diffusion	(19)
Δ^*	= dimensionless deposition, incorporating varying terminal velocities	(24)
ϵ	= coefficient depending on atmospheric stability	(51)
η	$= (1 + \alpha)ku^*$ = diffusion velocity for adiabatic atmosphere	(8)

		Equation
η_o	$= (1 + a)k_1\sigma$	(25)
η_β	$= (1 + a)k\epsilon u^*$ = diffusion velocity for diabatic atmosphere	(54)
Λ^*	= ratio of D_m predicted by (21) or (30), which neglects atmospheric diffusion to that predicted by (27), the analytic approximation.	(40)
$\overline{\Lambda}^*[\widetilde{\Lambda}^*]$	= ratio of D_m predicted by (10), which neglects dispersion of terminal velocities to that predicted by the analytic approximation.	(45), [(46)]
μ	= mean value of logarithm of terminal velocities	(13)
ν	= standard deviation of logarithm of terminal velocities	(13)
σ	= standard deviation of terminal velocities = characteristic fluctuating velocity	(14)
$\sigma/\overline{\Omega}$	= coefficient of variation of terminal velocities	(14)
ϕ	$= \ln(\widetilde{\Omega}/\eta)$	(16)
ω	= instantaneous velocity of particles with respect to air	(1)
Ω	= terminal velocity $= -\overline{\omega} > 0$	(4)
$\overline{\Omega}$	= mean value of terminal velocities	(14)
$\widetilde{\Omega}$	= median of terminal velocities	(14)
Ω^*	$= (\eta\overline{\Omega} + \eta_0\Omega_0)/(\eta + \eta_0)$	(22), (36)
Ω_0	$= k_o\overline{\Omega}$	(28), (34)

$\overline{}$ Mean value, expected value

$'$ Deviation from mean, e.g. $\omega' = \omega - \overline{\omega}$

$\sim$ Median value

Abstract

When a cluster composed of particles of varying terminal velocity is released in the atmosphere, at least two important physical processes contribute to the subsequent separation of these particles. The first is separation of the particles due to small scale turbulence; the second is separation due to the varying terminal velocities of the particles. It is shown that a term in the equation of turbulent diffusion, normally neglected, represents the contribution to the separation due to variation in terminal velocity. This term can be transformed by conventional mixing length assumptions and the resulting differential equation may be solved analytically for an adiabatic atmosphere provided it is assumed that the added term is independent of the x coordinate. This solution differs from that obtained by Rounds (or Bosanquet) for the case of monodisperse particulate diffusion in an adiabatic atmosphere only in that the diffusion coefficient is altered by an amount which depends on the statistical parameters of the distribution of terminal velocities.

The parameters are worked out in detail for a lognormal distribution of terminal velocities. By reference to a symbolic integration of the basic monodisperse solution it is shown that, in the surface layer the pattern of non-dimensional ground deposition depends only on the logarithm of the ratio of the median terminal velocity to the friction velocity, and on the coefficient of variation of the terminal velocity distribution. Comparison of the analytic solution for the polydisperse cloud with exact numerical integration of the monodisperse solution indicates the error in the analytic solution is tolerable for a

wide range of the basic parameters despite the neglect of the x variation of the increment in diffusion coefficient due to the distribution of terminal velocities.

The analytic polydisperse solution is then applied to a diabatic atmosphere (after Godson) and criteria are developed to indicate when neglect of atmospheric diffusion or particle terminal velocity distribution is permissible. For the range of particle sizes of major interest in air pollution studies, it is shown that normal variations in atmospheric stability and wind speed are sufficient to make neglect of either factor invalid for a substantial number of occasions.

1. Introduction

When a cluster composed of particles of varying sizes and terminal velocities is released in the atmosphere, at least two important physical processes contribute to the subsequent separation of these particles. The first is simply separation of the particles due to eddy diffusion; the second source of separation is due to the varying terminal velocities of the particles comprising the diffusing cluster. The relative importance of each of these processes obviously depends on the statistical distribution, in particular the mean and standard deviation, of the terminal velocities, and the diffusive power of the atmosphere as expressed by some characteristic vertical diffusion velocity.

It is the purpose of the present paper to consider the relative importance of each of these processes and to develop general criteria for permissible neglect of either one or the other. It is convenient to adopt a mathematical model of diffusion from a continuous infinite elevated line source as developed by Rounds (1955). This treatment is mathematically valid for the diffusion of monodispersed particles in the surface layers of an adiabatic atmosphere only. The line source limitation is not too serious since the terminal velocities of the particles act in the vertical plane. The restriction to the adiabatic atmosphere is more serious, but Godson (1958) has developed an approximate method of applying the original Rounds solution to the diabatic atmosphere.

The method of procedure will be to consider the mathematical solution for a monodispersed cloud and then to integrate this solution over a distribution of particle sizes. The integration can be performed

by numerical methods only, but the value of the integral is a function only of a virtual diffusion velocity and the median and standard deviation of the particle terminal velocities. A method of incorporating the effect of varying terminal velocity directly into the diffusion equation is then developed, and it is shown that the resulting analytical solution is a good approximation to the numerical integration. Criteria, based on the analytical solution, are then developed, in order to assess the relative importance of the particle size distribution on the deposition patterns under specified terrain and wind speed.

2. Diffusion of monodispersed particles in an adiabatic atmosphere

It is of some importance in what follows to consider the general equation of turbulent diffusion for particles of uniform terminal velocity. The instantaneous flux of particles will then be

$$F = (V + \omega)c \quad , \tag{1}$$

where V is the instantaneous vector velocity of the air, ω is the instantaneous velocity of the particle with respect to the air, and c is the instantaneous concentration of the diffusing particles. The equation of continuity is now

$$\frac{\partial c}{\partial t} = - \operatorname{Div} F \; . \tag{2}$$

Upon averaging both sides of (2) and employing the usual Reynold's averaging postulates, one obtains

$$\frac{\partial \overline{c}}{\partial t} + \operatorname{Div}((\overline{V} + \overline{\omega})\overline{c}) = - \operatorname{Div}(\overline{V'c'} + \overline{\omega'c'}) \; , \tag{3}$$

where $\omega' = \omega - \overline{\omega}$, $c' = c - \overline{c}$, and $V' = V - \overline{V}$.

Rewriting (3) to apply to the steady state solution from an infinite line source oriented normal to the direction of the mean wind, $\bar{u}$, we obtain

(4) $$\bar{u}(z)\frac{\partial \bar{c}}{\partial x} - \Omega \frac{\partial \bar{c}}{\partial z} = -\,\mathrm{Div}\,\overline{(w' + \omega')c'} \quad ,$$

where w is the vertical component of atmospheric velocity.

As is customary, the term Div $\overline{u'c'}$ has been neglected in comparison with the term $\bar{u}\,\partial\bar{c}/\partial x$. For a monodispersed particle, the term $\bar{c}\,\partial\Omega/\partial z$ can be important only near the release line since this term is non-zero only for the time or vertical distance required for the particles to achieve terminal velocity. On the scale of interest here, this term is negligible. For convenience $-\bar{\omega}$ has been replaced by Ω, the terminal velocity of the diffusing particles. Since the particles are heavier than air $\bar{\omega}$ will always be negative, and Ω will therefore be a positive quantity. In the term $\overline{\omega'c'}$ on the right hand side of (4), ω' represents fluctuating departures from particle terminal velocity. For monodispersed particles, such departures are due to inertia and reflect the inability of a particle to adjust instantaneously to the fluctuating velocity field. Evaluation of this term obviously depends on the previous history of the particle. It is customary to neglect $\overline{\omega'c'}$ in comparison with the other term involving w but as we shall see later, this term becomes quite important when there is a large variation of terminal velocity amongst the diffusing particles.

The right hand side of (4) is transformed by conventional mixing length assumptions, i.e., $c' = -\ell\,\partial\bar{c}/\partial z$, $\omega' = u^*$, $K = u^*\ell$ where u^* is a characteristic fluctuating velocity and ℓ is a characteristic scale of vertical velocity fluctuations, whence

$$\bar{u}(z)\,\frac{\partial \bar{c}}{\partial x} - \Omega\,\frac{\partial \bar{c}}{\partial z} = \frac{\partial}{\partial z}\,\overline{(u^* + \omega')\,\ell}\,\frac{\partial \bar{c}}{\partial z} \approx \frac{\partial}{\partial z}\,K\,\frac{\partial \bar{c}}{\partial z}\,, \tag{5}$$

where for the moment $\omega'\ell$ has been set equal to zero. In the surface layers of an adiabatic atmosphere $\ell = kz$ and u^* is constant with height whence

$$\left\{\begin{aligned} &K = u^* kz \\ &\frac{\bar{u}(z)}{u^*} = \frac{1}{k}\,\ell n(z/z_0) \approx \frac{1}{k}\,\ell n\left(\frac{h}{z_0}\right)\left(\frac{z}{h}\right)^{a} \end{aligned}\right. \quad , \tag{6}$$

where u^* is the constant friction velocity, equal to the square root of the surface stress divided by the density; h is a reference height, identified as the height of the source in what follows; and k is von Karman's constant ≈ 0.4. The approximation to the logarithmic wind profile was introduced by Calder (1949), and employed by Deacon (1949), Rounds (1955), F. B. Smith (1957), and others. The parameter a can be determined as a function of z_0, the roughness parameter, and the depth of the layer concerned while u^* can be evaluated from the logarithmic law provided $\bar{u}(h)$ and z_0 are known.

Rounds succeeded in solving (5) with the coefficients $\bar{u}$ and K given by (6) for the concentration of diffusing particles as a function of x and z. According to this diffusion model the flux through the plane $z = 0$ is simply

$$\Omega\, c(x, 0) \quad ,$$

where $c(x, 0)$ is the concentration at $z = 0$. This quantity is interpreted as the deposition on a flat surface at $z = 0$. The deposition at $z = 0$, from a line source at height h, according to Rounds may then be written

$$\Delta = \frac{fD}{Q} = \frac{p}{\Gamma(1+p)} \left(\frac{f}{x}\right)^{1+p} e^{-\frac{f}{x}}, \tag{7}$$

where D is the deposit [gm/(m^{2}sec)], Q is the source strength [gm/(m sec)], Γ is the gamma function, Δ is the dimensionless deposition, and

$$\left\{\begin{aligned} p &= \frac{\Omega}{(1+a)ku^*} = \frac{\Omega}{\eta} \\ f &= \frac{h\,\bar{u}(h)}{(1+a)^2 ku^*} = \frac{H}{\eta} = \frac{h\,\ell n(h/z_0)}{(1+a)^2 k^2} \end{aligned}\right. \tag{8}$$

The quantity η has the dimensions of a velocity and will be referred to hereafter as the diffusion velocity. The dimensionless parameter p therefore represents the ratio of particle terminal velocity to atmospheric diffusion velocity. The dominant length scale is f which (by (6)) is the height of the source scaled by the ratio of the vertically averaged wind speed (determined from the ground to h) to the diffusion velocity. Since the diffusion velocity is proportional to the wind speed, f can be expressed as a function of wind speed, wind shear and surface roughness only. It will be noted that f is independent of the particle terminal velocity and depends only on source height and atmospheric wind and diffusion parameters.

It may be verified that the integral of (7) with respect to x from 0 to ∞ is the source strength Q and hence all the material released is eventually deposited on the ground. In particular the maximum deposit occurs at

$$x_m = \frac{f}{1+p} = \frac{h\,\bar{u}(h)}{(1+a)[(1+a)ku^* + \Omega]} = \frac{H}{\eta+\Omega}, \tag{9}$$

and clearly demonstrates that the point of maximum concentration moves closer to the source as the terminal velocity, (Ω), or as the diffusion

velocity (η) increases. It is clear from (9) that insofar as the location of the maximum deposit is concerned, the diffusion and terminal velocities enter as a linear combination. When $\eta >> \Omega$, the effect of the terminal velocities of the particles is negligible. Similarly, the effect of diffusion on the location of the point of maximum concentration becomes negligible when $\Omega >> \eta$. Since $(1 + \alpha)k$ is of the order of 1/2, the two effects are about equally important when $u^* \approx 2\Omega$.

The value of the maximum deposit, D_m, [Δ_m, in dimensionless units] is given by

$$\Delta_m = \frac{fD_m}{Q} = \frac{p}{\Gamma(1 + p)}\left(\frac{1 + p}{e}\right)^{1+p} \quad . \tag{10}$$

From (8) it is apparent that the maximum non-dimensional deposit depends only on the ratio of the terminal velocity to the diffusion velocity. Restoring dimensions to (10) we have

$$\frac{D_m}{Q} = \frac{(1 + \alpha)\Omega}{h\,\overline{u}(h)\,\Gamma(1 + p)}\left(\frac{1 + p}{e}\right)^{1+p} , \tag{11}$$

showing that the value of the maximum deposit is proportional to the terminal velocity and inversely proportional to the wind speed and height of release.

3. Diffusion of polydispersed clouds

3.1 Integration of monodisperse solution

The above discussion indicates that there are circumstances when diffusion is relatively unimportant compared to the terminal velocity of the diffusing particles and vice-versa. We now investigate the situation when the diffusing cloud is composed of particles with a distribution of terminal velocities. Let the distribution of mass as a

function of particle terminal velocity be denoted by $Q(\Omega)$. The function $Q(\Omega)$ has the property

$$\int_0^\infty Q(\Omega)\, d\Omega = Q .$$

If it is assumed that particles with different terminal velocities diffuse independently of each other, the deposition pattern would be

$$\Delta = \frac{fD}{Q} = \int_0^\infty \frac{Q(\Omega)p}{Q\Gamma(1+p)} \left(\frac{f}{x}\right)^{1+p} e^{-\frac{f}{x}}\, d\Omega . \tag{12}$$

Since p is a function of Ω, (12) cannot be integrated analytically for any reasonable $Q(\Omega)$.

We consider the case now where the distribution of mass is lognormal in terminal velocity. Since negative terminal velocities $(-\Omega)$ are excluded, the lognormal distribution is a reasonable one which may be expected to arise in practice. For this type of distribution

$$Q(\Omega) = \frac{Q}{\sqrt{2\pi}\,\nu\Omega} \exp\{-[\ln\Omega - \mu]^2/2\nu^2\} \quad , \tag{13}$$

where μ is the mean value of the logarithm of the terminal velocities, and ν is the logarithmic standard deviation. The arithmetic average, $\bar{\Omega}$, the median, $\tilde{\Omega}$, the arithmetic standard deviation, σ, and the coefficient of variation $\sigma/\bar{\Omega}$ are given by

$$\left\{\begin{array}{ll} \bar{\Omega} = e^{\mu+\nu^2/2} \quad , & \tilde{\Omega} = e^{\mu} \\ \sigma = \bar{\Omega}[e^{\nu^2} - 1]^{1/2} \quad , & \dfrac{\sigma}{\bar{\Omega}} = (e^{\nu^2} - 1)^{1/2} \end{array}\right. \tag{14}$$

Upon substituting (13) into (12) and with the help of (8) changing the variable of integration to p, (12) becomes

(15) $$\Delta = \frac{fD}{Q} = e^{-\frac{f}{x}} \int_0^\infty \frac{\exp[-(\ln p - \phi)^2/2\nu^2]}{\sqrt{2\pi}\,\nu\,\Gamma(1+p)} \left(\frac{f}{x}\right)^{1+p} dp \ ,$$

where

(16a) $$\phi = \mu - \ln\eta = \ln\left(\frac{\tilde{\Omega}}{\eta}\right) \ ,$$

and inversely,

(16b) $$\eta = e^{\mu - \phi} \ .$$

In terms of the non-dimensional variables employed above, the required integral will be a function of x/f, ϕ, and ν only. The parameter ϕ represents the logarithm of the ratio of the mass-median terminal velocity to the diffusion velocity. A large negative value of ϕ implies that the diffusion velocity is large compared to the median terminal velocity of the diffusing particles while a positive value of ϕ indicates that the median terminal velocity is greater than the diffusion velocity. In effect, ϕ estimates the relative importance of terminal velocity and diffusion velocity on the integrated pattern of deposition. As far as the non-dimensional pattern of deposition is concerned, atmospheric diffusion enters only in the value of ϕ. Consequently if the logarithm of the diffusion velocity η is very small compared to the logarithm of the median particle terminal velocity, the pattern of non-dimensional deposition is almost independent of the atmospheric diffusion velocity. The deposition pattern for this case will then depend only on $\tilde{\Omega}$ and ν. It should be noted that since $f = H/\eta$, the diffusion velocity appears in the denominator of the left hand side of (15) so that even when η is very small compared to $\tilde{\Omega}$, the non-dimensional value of the deposition will be proportional to η. However, the shape of the curve of deposition will not depend on η.

The parameter ν is the only other independent parameter on which (15) can depend. For the case $\eta << \tilde{\Omega}$, it is clear that the pattern of non-dimensional deposition will depend only on $\tilde{\Omega}$ and ν. The parameter ν represents the logarithmic standard deviation of the particle terminal velocities. A more meaningful interpretation of ν emerges from (14), namely the coefficient of variation, $\sigma/\overline{\Omega}$, of the terminal velocity distribution is uniquely fixed by the value of ν. It is this latter interpretation which will be adopted hereafter. For very small values of ν, it seems clear that (15) should reduce to (7) since the terminal velocity dispersion around the mean value will then be quite small. From these considerations it appears that under certain circumstances, i.e. when $\eta << \tilde{\Omega}$, atmospheric diffusion is not an important factor in describing the pattern of ground deposition while for other cases, i.e. when ν or $\sigma/\overline{\Omega}$ is small, the spread of terminal velocity will not be important. It is the purpose now to develop quantitative criteria for neglecting either one or the other effect.

3.2 Deposition from polydispersed cloud, neglecting diffusion

The presence of ν in the exponential term of (15) indicates that the effect of varying terminal velocity is to spread the particles much like a diffusion process might spread particles of uniform terminal velocities. This is most easily seen when one considers the ground deposition distribution pattern resulting from release of particles whose terminal velocities vary like (13), in a zero diffusion atmosphere. For this case, [neglecting η in (9)] a particle with terminal velocity Ω_1, will strike the ground at

(17) $$x = H/\Omega_1 \, .$$

This result may be obtained by equating the right hand side of (4) to zero, and letting $\Omega = \Omega_1$ whence, using (6) we have

(18) $$x = \int_0^h \frac{\bar{u}(z)}{\Omega_1} dz = \frac{1}{k}\int_0^h \frac{u^*}{\Omega_1} \ln\left(\frac{h}{z_o}\right)\left(\frac{z}{h}\right)^a dz$$

$$= \frac{h\bar{u}(h)}{(1+a)\Omega_1} = \frac{H}{\Omega_1} \, .$$

This expression, in which the distance downwind traveled by the particle is inversely proportional to its terminal velocity was used by Trouern-Trend and J. Monaghan (1956) and in a U. S. Discussion paper presented at the 12th Tripartite Conference (1957).

For a continuous line source, the deposit per m^2 per sec at $z = 0$ as a function of x, resulting from the lognormal distribution of mass with terminval velocity (13) is

(19a) $$D_o = \frac{Q}{\sqrt{2\pi}\,\nu x} \exp\left\{-\left[\ln\left(\frac{H}{x}\right) - \mu\right]^2/(2\nu^2)\right\}$$

or

(19b) $$\Delta_o = \frac{fD_o}{Q} = \frac{f/x}{\sqrt{2\pi}\,\nu} \exp\left\{-\left[\ln\left(\frac{f}{x}\right) - \phi\right]^2/(2\nu^2)\right\} \, ,$$

where the zero subscript indicates deposition for zero atmospheric diffusion conditions. If the terminal velocity of the particles were constant, all the particles would strike the ground at the point predicted by (17). The effect of the distribution of terminal velocities is to spread the ground deposition over a wide area. The effect is very much like a diffusion process, even though in this case atmospheric diffusion has been assumed to be zero. The distribution predicted by (19) has a maximum at

(20) $$x_m = \frac{H}{\bar{\Omega}\, e^{\frac{1}{2}\nu^2}} \quad \text{or} \quad \frac{x_m}{f} = e^{-\phi - \nu^2} \quad ,$$

and the value of the maximum deposit is given by

(21) $$\frac{D_m}{Q} = \frac{\bar{\Omega}}{\sqrt{2\pi}\ \nu H} \quad \text{or} \quad \Delta_m = \frac{fD_m}{Q} = \frac{1}{\sqrt{2\pi}\ \nu}\, e^{\phi + \frac{1}{2}\nu^2} \quad ,$$

where $\bar{\Omega}$ is defined by (14).

3.3 Analytic approximation for polydispersed clouds

Since the varying terminal velocity acts as a diffusing agent, it seems desirable to incorporate this effect into the diffusion equation. The spreading effect of the varying terminal velocity is implicit in the term $\overline{\omega' c'}$ of equation (4). If the diffusing particles have widely varying terminal velocities, ω' can now be interpreted as the deviation of the velocity of the particle from the average terminal velocity of all the particles in a given volume.

Depending on the relative magnitude of ν and μ, ω' can easily take on values as great as or greater than w'. For this case we consider the more complete diffusion equation,

(22) $$\bar{u}\,\frac{\partial \bar{c}}{\partial x} - \Omega^*\,\frac{\partial \bar{c}}{\partial z} = \frac{\partial}{\partial z}\,\overline{[u^* + \omega']\ell}\,\frac{\partial \bar{c}}{\partial z} \quad .$$

We now identify ω' as a characteristic fluctuating terminal velocity which is proportional to the arithmetic standard deviation of the terminal velocity distribution. Thus

(23) $$\overline{\omega' \ell} = k_1 \sigma z\,\frac{\partial \bar{c}}{\partial z} \ ,$$

where k_1 is a constant. Thus far Ω^* in equation (22) has not been defined but it should be proportional to the arithmetic average of the terminal velocities. The constant of proportionality will be established

below. It should be noted at this point that since the heavier particles fall out near the origin, the effective $\overline{\Omega}$ and σ undoubtedly decrease with x; it is not certain how the ratio $\sigma/\overline{\Omega}$ behaves as a function of distance downwind. In any event, for mathematical feasibility we neglect the variation of $\overline{\Omega}$ and σ with x, with the hope that adjustment of the two constants will grossly take into account the effect of this variation.

By using (23) in (22) it can be seen that (22) is formally the same as (5) or

$$\overline{u}(z)\,\frac{\partial\overline{c}}{\partial x} = \frac{\partial}{\partial z}\left[\left(ku^* + k_1\sigma\right) z\,\frac{\partial\overline{c}}{\partial z} + \Omega^*\overline{c}\right] .$$

The only difference is that the diffusion coefficient has been increased by an amount proportional to the arithmetic standard deviation of the terminal velocities. The solution will therefore have the same form as (7) and may be written

$$(24) \qquad \Delta^* = \frac{f^*D}{Q} = \frac{p^*}{\Gamma(1+p^*)}\left(\frac{f^*}{x}\right)^{1+p^*} e^{-\frac{f^*}{x}} ,$$

with

$$f^* = \frac{H}{(1+a)(ku^* + k_1\sigma)} = \frac{H}{\eta + \eta_0}$$

$$p^* = \frac{\Omega^*}{(1+a)(ku^* + k_1\sigma)} = \frac{\Omega^*}{\eta + \eta_0} ,$$

where, for typographical convenience, we have let

$$(25) \qquad \eta_0 = (1+a)k_1\sigma .$$

Differentiating (24) and changing to the dimensionless units Δ and x/f of (7), we have

$$\frac{x_m}{f} = \frac{f^*}{f(1 + p^*)} \tag{26}$$

and

$$\Delta_m = \frac{fD_m}{Q} = \frac{f\,p^*}{f^*\Gamma(1 + p^*)}\left(\frac{1 + p^*}{e}\right)^{1+p^*} . \tag{27}$$

It remains now to fix the constants k_1 and Ω^*. This is done most simply by considering the solution of (22) for zero diffusion conditions, the case for which (19) is formally valid. For $u^* = 0$, the solution is (24) with $\eta = 0$. Thus

$$\Delta_0 = \frac{f_0 D_0}{Q} = \frac{p_0}{\Gamma(1 + p_0)}\left(\frac{f_0}{x}\right)^{1+p_0} e^{-\frac{f_0}{x}} , \tag{28}$$

$$f_0 = \frac{H}{\eta_0} ,$$

$$p_0 = \frac{\Omega_o}{\eta_0} = \frac{k_o\overline{\Omega}}{\eta_0} ,$$

where the zero subscript indicates a zero diffusion solution. In particular the point of maximum concentration and the value of the maximum concentration for $\eta = 0$ is

$$\frac{x_m}{f} = \frac{f_o}{f(1 + p_o)} \tag{29}$$

and

$$\Delta_m = \frac{fD_m}{Q} = \frac{f}{f_o}\frac{p_o}{\Gamma(1 + p_o)}\left(\frac{1 + p_o}{e}\right)^{1+p_o} . \tag{30}$$

It is clear that since the variation of Ω_0 and σ with x has been neglected, (28) cannot be identical with (19), but since we have two undetermined constants at our disposal we can force (28) to reproduce the location

and value of the maximum concentration predicted by (19). By equating (29) and (20), we obtain

$$\eta_0 + \Omega_0 = \frac{\Omega_0}{p_0} + \Omega_0 = \Omega_0\left(\frac{1+p_0}{p_0}\right) = \eta_0(1+p_0) = e^{\mu+\nu^2} \tag{31}$$

and by equating (30) and (21) we have

$$\frac{\Omega_0}{\Gamma(1+p_0)}\left[\frac{1+p_0}{e}\right]^{1+p_0} = \frac{1}{\sqrt{2\pi}\,\nu}\, e^{\mu+\frac{1}{2}\nu^2} \tag{32}$$

It follows from (14), (28), and (31) that

$$1 + p_0 = \frac{e^{\frac{1}{2}\nu^2}}{(1+\alpha)k_1(e^{\nu^2}-1)^{\frac{1}{2}}} \tag{33}$$

$$k_0 = \frac{\Omega_0}{\overline{\Omega}} = e^{\frac{1}{2}\nu^2} - k_1(1+\alpha)(e^{\nu^2}-1)^{\frac{1}{2}} \tag{34}$$

Using (31) in (32), we also have

$$\frac{p_0/(1+p_0)}{\Gamma(1+p_0)}\left[\frac{1+p_0}{e}\right]^{1+p_0} = \frac{1}{\sqrt{2\pi}\,\nu}\, e^{-\nu^2/2} \tag{35}$$

It is evident that p_0, $(1+\alpha)k_1$ and $\Omega_0/\overline{\Omega}$ are functions of ν only. Equation (35) is relatively easy to solve for $1+p_0$ as a function of ν. The solution is graphed in Figure 1 as a function of ν. It may be seen that p_0 approaches zero as ν approaches infinity and becomes quite large as ν becomes small. Since p_0 contains the diffusion term due to terminal velocity dispersion in the denominator, it is clear from Figure 1 that small values of p_0 imply a relatively large diffusion constant for large values of ν which is a quite reasonable result.

Although not necessary in what follows, it is interesting to note the behavior of $(1+\alpha)k_1$ and $k_0 = \Omega_0/\overline{\Omega}$ as functions of ν. This is shown

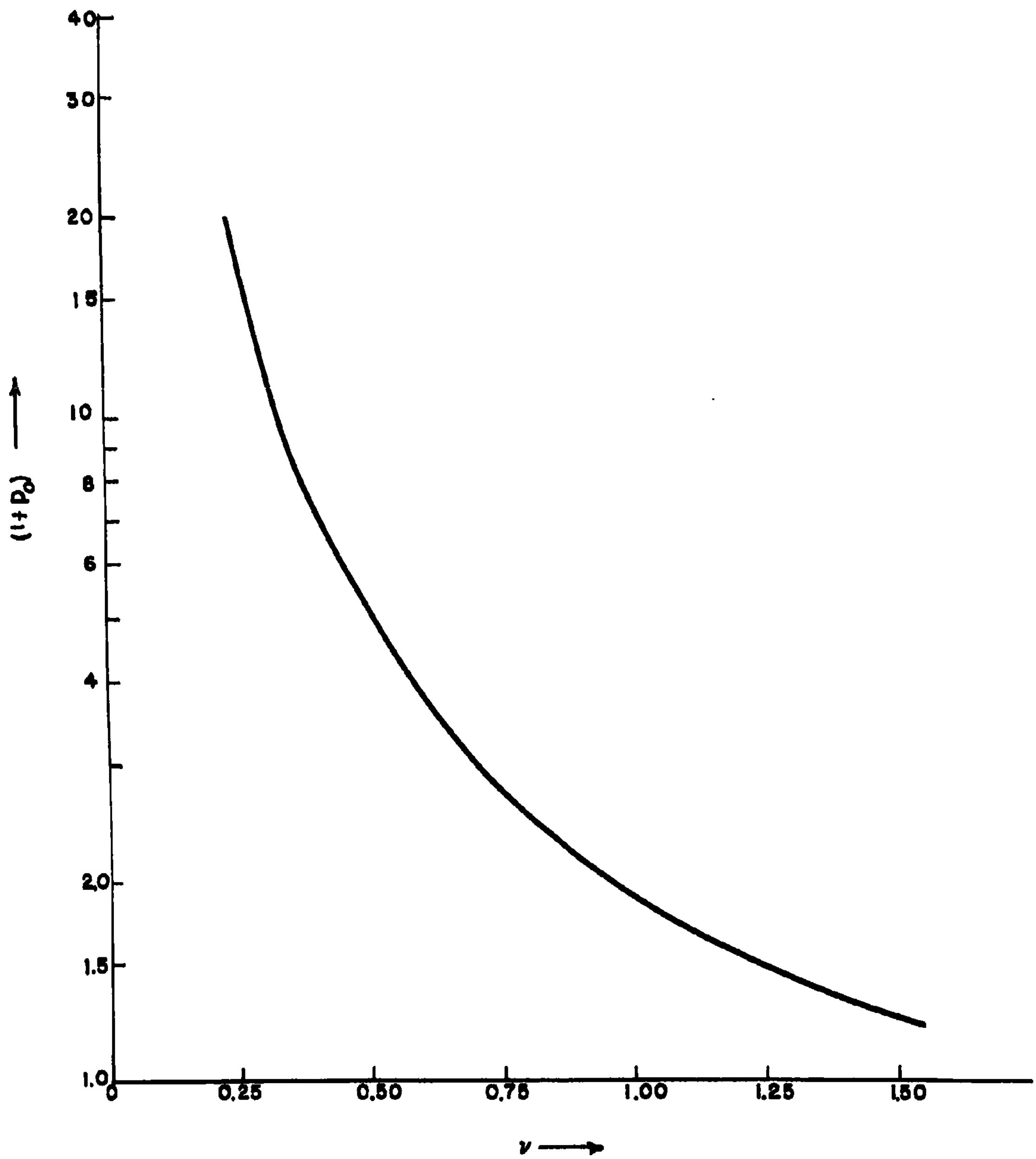

Figure 1. $(1 + p_o)$ as a function of ν .

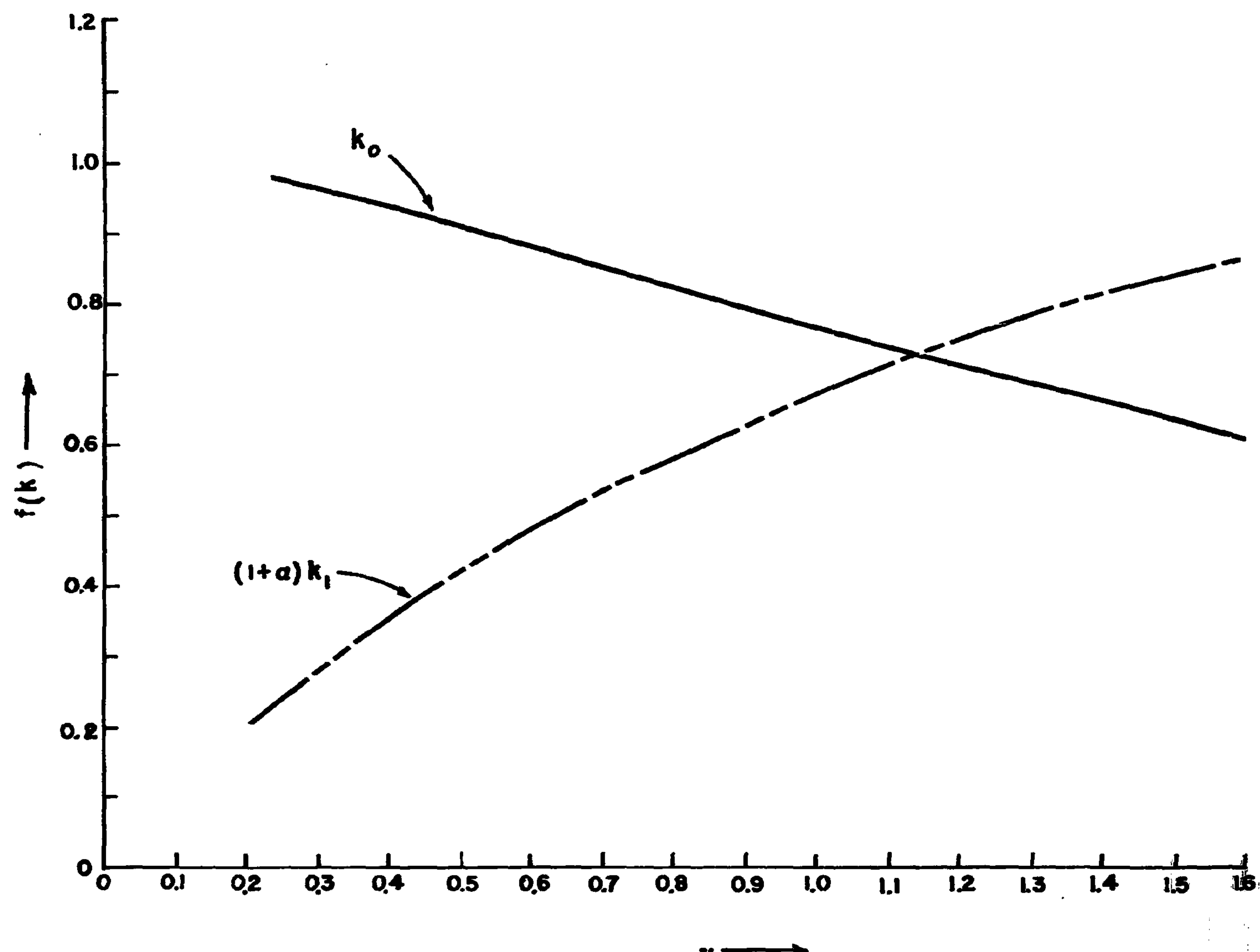

Figure 2. $(1 + \alpha)k_1$ and k_0 as functions of ν.

in Figure 2 and it is at once apparent that $(1 + \alpha)k_1$, which may be regarded as a weighting function for the arithmetic standard deviation of the particle terminal velocity, becomes large as ν becomes large. For small ν, $(1 + \alpha)k_1$ approaches zero and $\Omega_0/\bar{\Omega}$ approaches 1 indicating that when ν is small the only important term is $\bar{\Omega}$.

The constants k_1 and k_0 have now been determined from the zero diffusion solution. Thus it remains to incorporate these constants into (24). It will be noted that Ω^* is still largely undefined. If we now consider the case where ν is very small, so that the spread of particle terminal velocities may be neglected, it seems reasonable to assume that

$$\Omega^* = \bar{\Omega} \quad \text{and} \quad p = \bar{\Omega}/\eta \,.$$

This procedure gives us the correct solution when there is zero dispersion of particle terminal velocity. It is desirable to define Ω^* so that it will approach $\bar{\Omega}$ for zero terminal velocity dispersion and will approach Ω_0 for zero atmospheric conditions. Such an expression is

$$\Omega^* = \frac{\eta\bar{\Omega} + \eta_0\Omega_0}{\eta + \eta_0} \,. \tag{36}$$

In terms of the statistics of the terminal velocity distribution and arranging terms so that the variables, ϕ and η of (15) and (16) are introduced, we have

$$\frac{\Omega^*}{\eta} = e^{\phi+\frac{1}{2}\nu^2}\left[\frac{p_0 e^{\nu^2} + (1 + p_0)^2\, e^{-(\phi+\frac{1}{2}\nu^2)}}{(1 + p_0)e^{\frac{1}{2}\nu^2} + (1 + p_0)^2\, e^{-(\phi+\frac{1}{2}\nu^2)}}\right]$$

$$p^* = \frac{(1 + p_0)^2\, e^{\phi+\frac{1}{2}\nu^2} + p_0\, e^{2(\phi+\nu^2)}}{\left(1 + p_0 + e^{\phi+\nu^2}\right)^2} \tag{37}$$

and

$$f^* = \frac{f(1 + p_0)}{1 + p_0 + e^{\phi+\nu^2}} \tag{38}$$

3.4 Comparison of analytic approximation with numerical integration and zero diffusion model

The solution (24) with the definitions (36), (37), and (38) constitute an analytical solution of the problem we posed initially. Several approximations have been made in the derivation, and it is desirable to compare the predictions of (24) with the results of the numerical integration of (15) which is certainly more exact. In fact what has been done thus far is simply to invent a diffusion process, based in part upon physical notions, which it is hoped will reproduce the main features of the numerical integration. Of course one could use, for Ω^* the median, mode or some other measure of central tendency. This would give similar but not identical results.

Accordingly, (15) was integrated numerically on an IBM 650 computer. The limits, zero to infinity, were replaced by $\underline{p} = e^{\phi-3\nu}$ and $\overline{p} = e^{\phi+3\nu}$. By trial and error it was found that dividing p into 25 equally spaced intervals was sufficient to insure adequate accuracy in the numerical results. The numerical integrations were performed for the following values of ϕ and ν:

ϕ	ν	Figure
-1.00	0.55 1.0	3
0	0.30, 0.55, 1.0	4
+0.81	0.57	5
+1.50	0.57	5
+1.61	0.53	6
+2.30	0.53	6

The numerical integrations are shown in Figures 3 through 6 as indicated above. For comparison, the analytic approximation, (24), with the coefficients (36), (37), and (38), is shown along with the zero diffusion solution, (19). The case $\phi = -1$ (Fig. 3) represents the solution when the atmospheric diffusion velocity is about 2.3 times greater than the average terminal velocity of the particles. It may be seen that for $\nu = 0.55$, the solution neglecting atmospheric diffusion overestimates the point of maximum concentration by a factor of about 3, and the value of the maximum concentration by a factor of about 1/3. On the other hand, the analytic approximation is within 10% of the location and value of the maximum concentration and fits the numerical integration curve reasonably well over the entire range of integration. It may also be seen that for the case $\nu = 1$, the analytic approximate underestimates the value of the maximum concentration by less than 10%. The solution neglecting atmospheric diffusion yields a reasonable value of the maximum concentration which is, however, displaced downwind by more than a factor of 2. Attention is also called to the fact that the analytic approximation is very close to the numerical integration past the point of maximum concentration where the solution neglecting atmospheric diffusion is particularly bad.

Figure 4 summarizes the computations for $\phi = 0$, that is the case where the atmospheric diffusion velocity η is equal to the median terminal velocity of the diffusing particles. It may be seen that the analytic approximation is quite satisfactory for all three values of ν. The solution neglecting atmospheric diffusion is quite bad for $\nu = 0.30$ and 0.55 and yields a more reasonable approximation when $\nu = 1$.

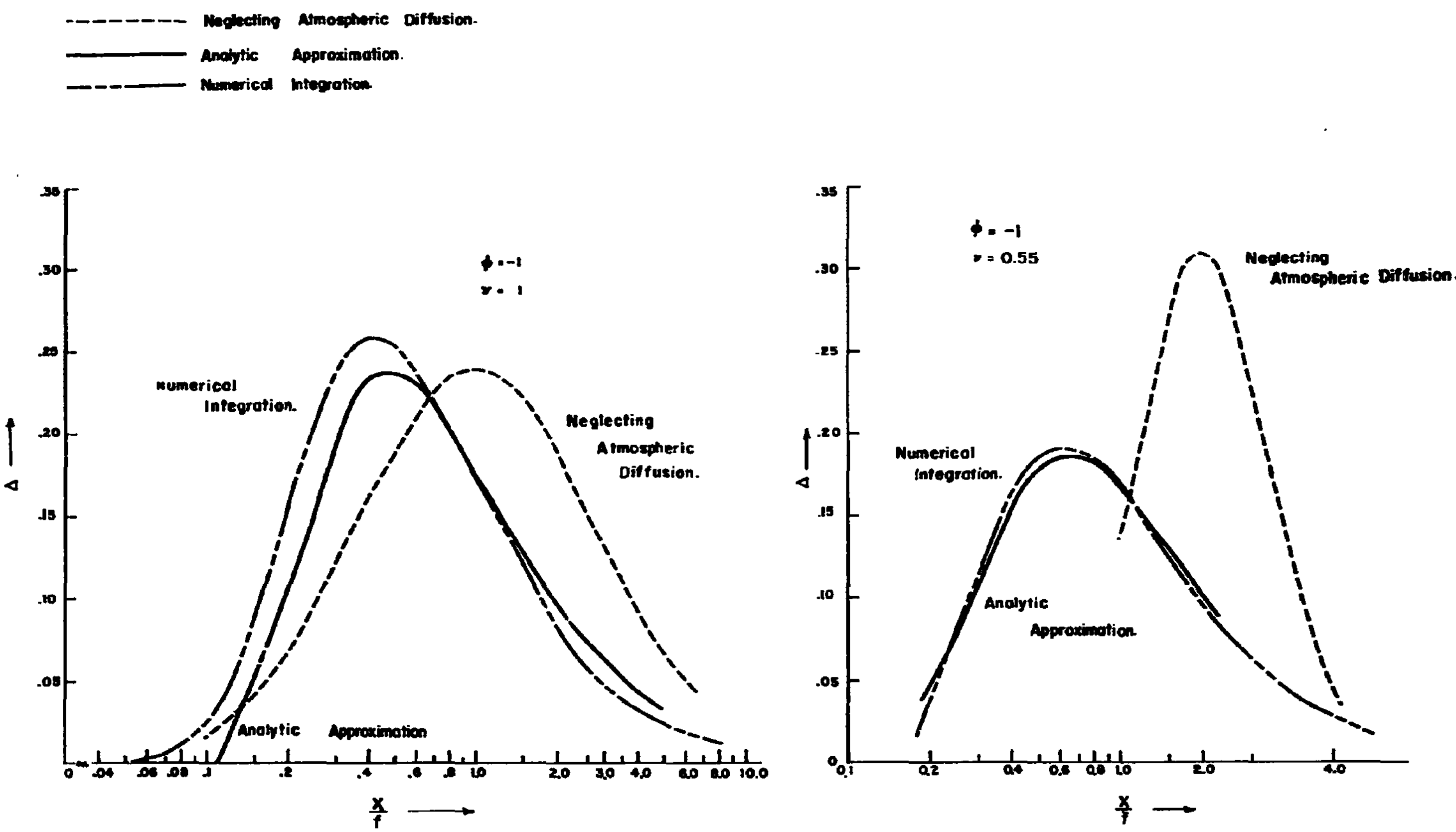

Figure 3. Distribution of non-dimensional deposit as a function of x/f for $\phi = -1$, $\nu = 0.55$ and 1.0 .

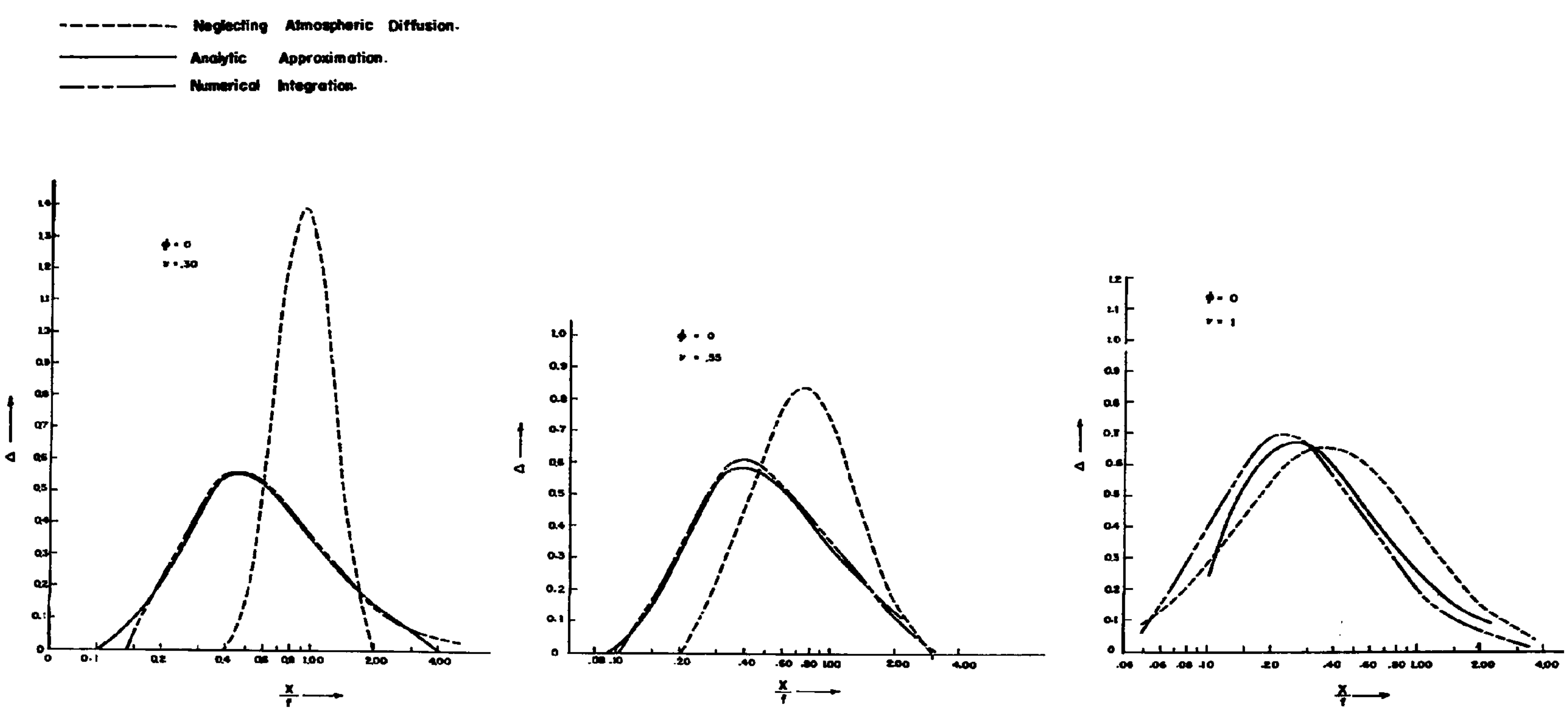

Figure 4. Distribution of non-dimensional deposit as a function of x/f for $\phi = 0$, $\nu = 0.3$, 0.55 and 1.0 .

Figure 5 summarizes the results for $\phi = 0.81$ and 1.50. Again the analytic approximation is quite close to the numerical integration while the solution, neglecting atmospheric diffusion, although closer to the numerical integration than was the case for small values of ϕ, is still not as good as the analytic approximation.

Finally, Figure 6 summarizes the results for $\phi = 1.61$ and 2.30. It will be noted that for these large values of ϕ (i.e., $\tilde{\Omega}$ equals about 5 or 10 times η), the solution neglecting atmospheric diffusion is tolerable for $\nu = 0.53$. The analytic solution, however, is still a better approximation even for these large relative values of $\tilde{\Omega}$.

In summary, the analytic approximation fits the more exact numerical integration reasonably well for all values of ϕ and ν investigated. It may be seen, however, that the solution neglecting atmospheric diffusion is quite bad for small values of ϕ and ν but becomes tolerable as ϕ and ν become large. This is not a surprising result since a large value of ϕ and ν imply that atmospheric diffusion processes are unimportant compared to the median terminal velocities and the spread of terminal velocities around the mean.

4. Relative importance of diffusion velocity and the spread of terminal velocity

4.1 When can atmospheric diffusion be neglected?

Now that the analytic approximation to the integral has been established to be reasonably accurate, it is possible to develop simple expressions for estimating under which circumstances atmospheric diffusion and/or variations in particle terminal velocity may be neglected. For example, equation (21) represents a solution in which atmospheric

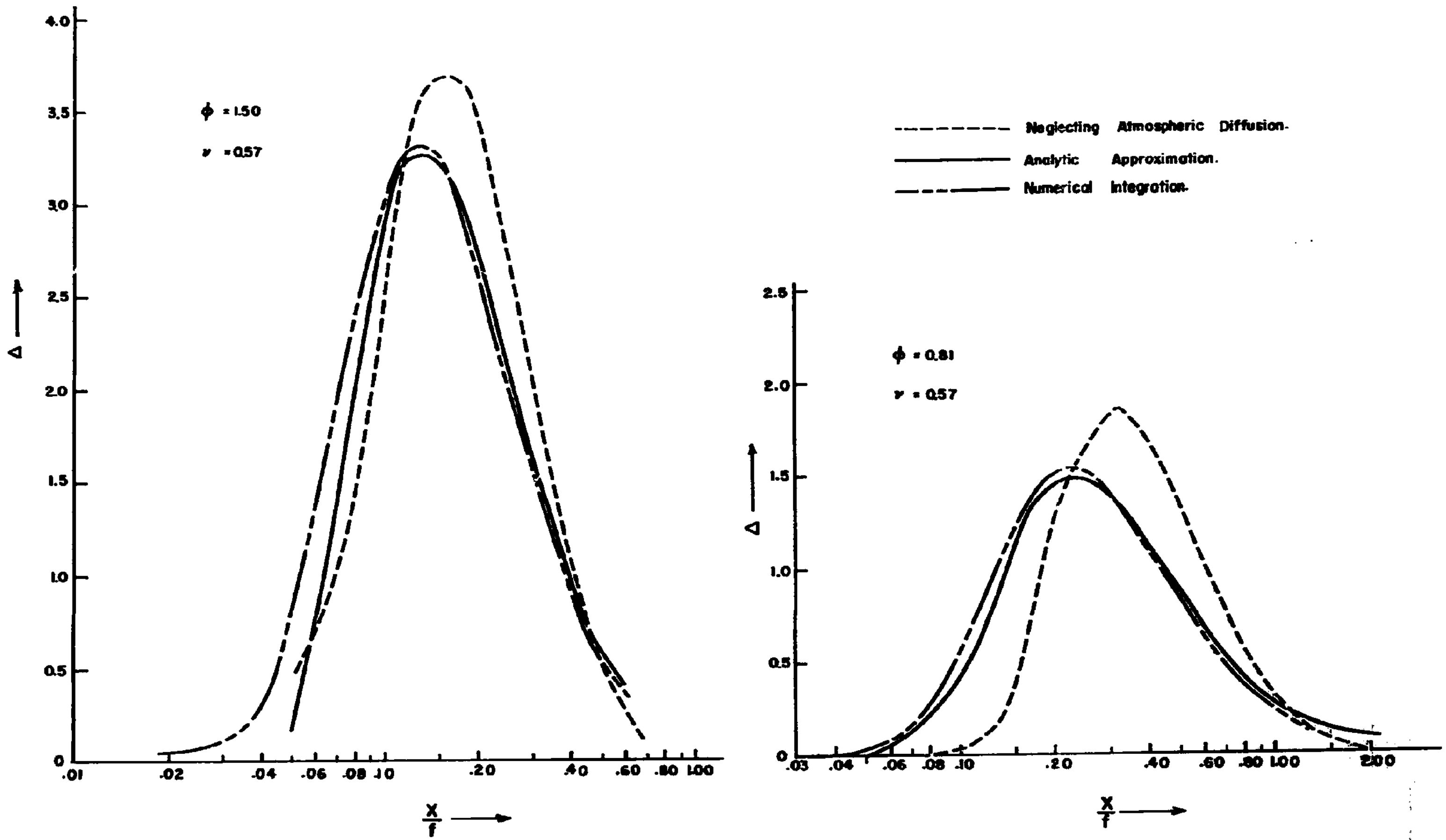

Figure 5. Distribution of non-dimensional deposit as a function of x/f for $\phi = 0.81$ and 1.50, $\nu = 0.53$.

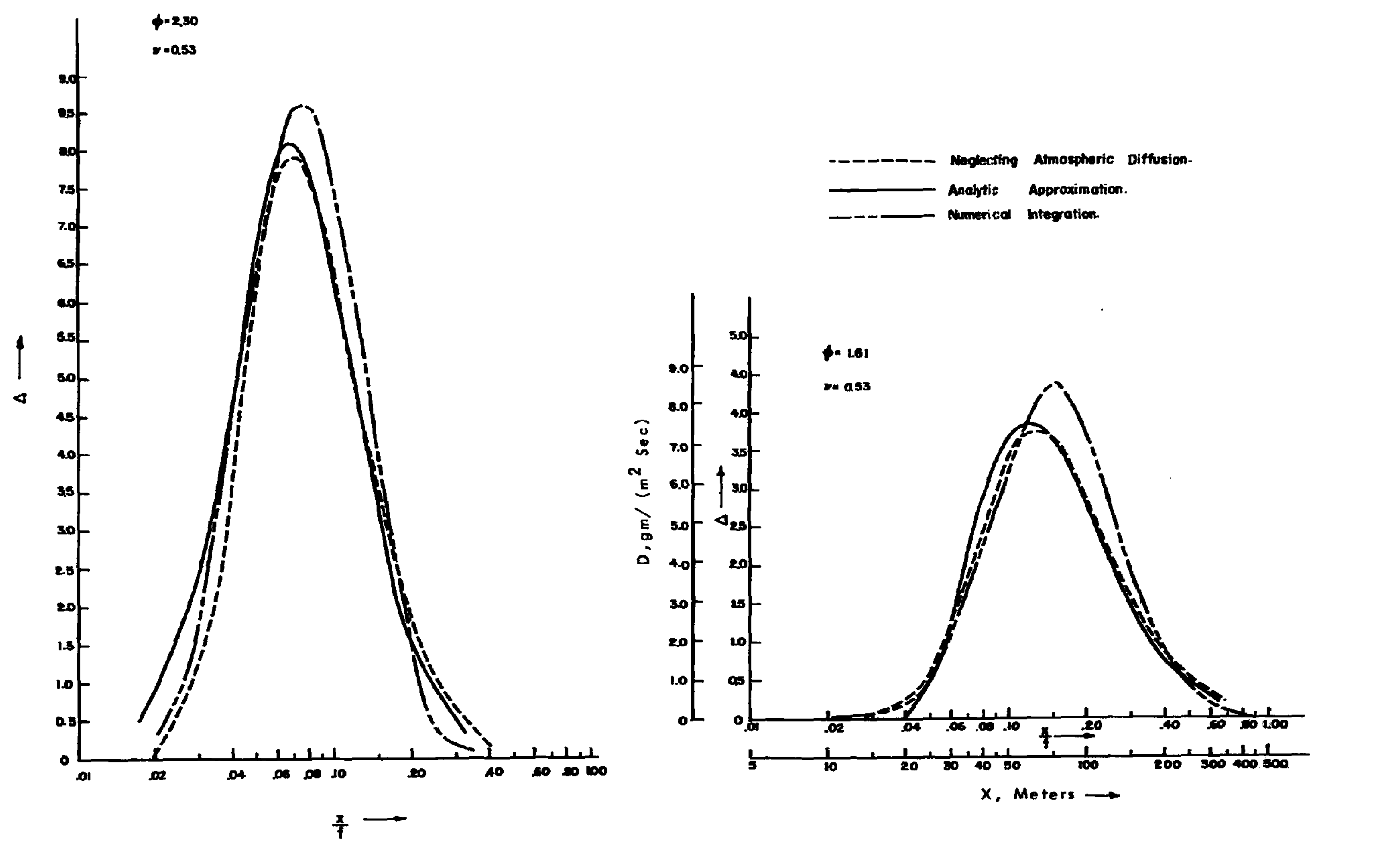

Figure 6. Distribution of non-dimensional deposit as a function of x/f for $\phi = 1.61$ and 2.30, $\nu = 0.53$. The auxiliary scale for the former yields the dimensional deposit as a function of distance downwind for $f = 511$ m and $\Omega = 1000$ gm/m/sec

diffusion is completely neglected. Under what circumstances is this neglect justified? The question is readily answered if we define a quantity R^* which is the ratio of the location of the maximum concentration predicted by equations (20) or (29) which neglects atmospheric diffusion to that predicted by (26) which includes the effect of both atmospheric diffusion and dispersion of particle terminal velocities. Similarly a quantity Λ^* relating the ratio of the predicted maximum deposit for the zero diffusion assumptions (21) or (30) to that predicted by (27). Thus

$$(39) \qquad R^* = \frac{(1+p^*)}{(1+p_0)}\frac{f_o}{f^*} = \frac{1+p^*}{1+p_0}\left[1 + (1+p_0)\, e^{-(\phi+\nu^2)}\right] \quad .$$

$$(40) \quad \Lambda^* = \frac{p_0}{p^*}\frac{\Gamma(1+p^*)}{\Gamma(1+p_0)}\left(\frac{1+p_0}{e}\right)^{1+p_0}\left(\frac{e}{1+p^*}\right)^{1+p^*}\left[1+(1+p_0)e^{-(\phi+\nu^2)}\right]^{-1} \quad .$$

Equation (39) is graphed in Figure 7. If R^* is close to unity there is little difference between the location of the maximum concentration predicted by either equation. It may be seen that neglect of the atmospheric diffusion term generally leads to an overestimate of the location of the maximum concentration. It may also be seen that for any value of ϕ, there always exists a ν for which atmospheric diffusion may be neglected with tolerable results. For example, the error in the location of the maximum concentration caused by neglect of atmospheric diffusion is 10% or less for $\phi > 2$ and $\nu > 0.9$; for $\phi = 0$, i.e., $\eta = \tilde{\Omega}$, the error is about 80% for $\nu = 0.55$, about 45% for $\nu = 1.0$ and about 12% for $\nu = 1.5$. Since the coefficient of variation of the distribution of mass as a function of terminal velocity

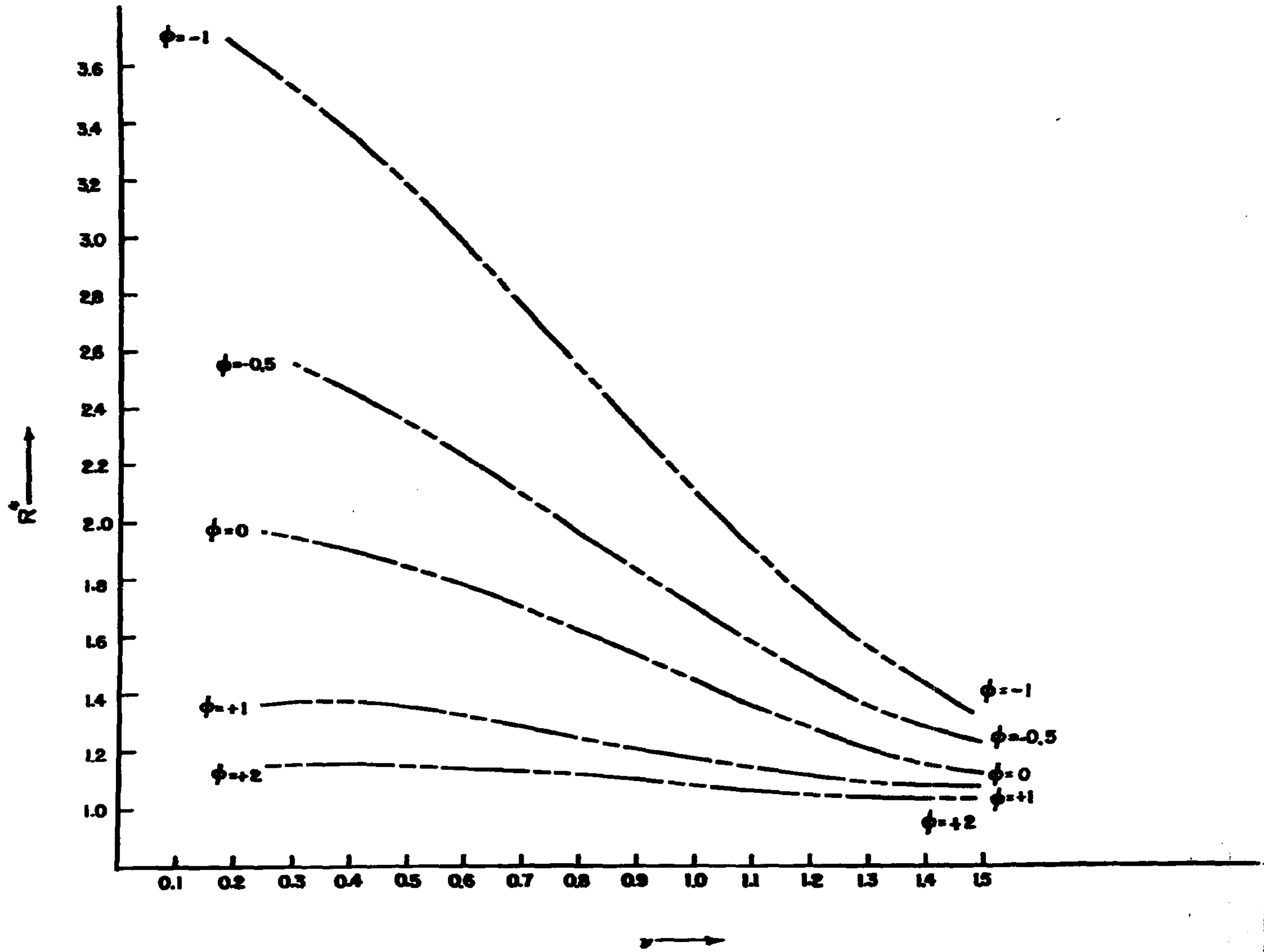

Figure 7. R^* as a function of ν for indicated values of ϕ .

is $(e^{\nu^2} - 1)^{1/2}$, a value of 1.5 for ν implies that $\sigma/\overline{\Omega} = 2.9$ which represents a very wide distribution of terminal velocity.

The curve of Λ^* as a function of ϕ and ν is shown in Figure 8. Again the zero diffusion solution badly overestimates the value of the maximum concentration for small values of ν. The percent error, however, becomes quite small (less than 10%) for $\nu = 0.65$, $\phi = -1.0$; $\nu = 0.77$ for $\phi = 0$, etc. In evaluating these numbers it should be borne in mind that we are dealing here with an infinite line source solution, and in general all errors in concentration would be roughly squared when applied to a point source.

From inspection of (39) and (40) it is apparent that since $p_0 \rightarrow p^* \rightarrow 0$ as $\nu \rightarrow \infty$, both R^* and $\Lambda^* \rightarrow 1$ for large values of ν. The rate at which both ratios approach one differs. For example, at $\nu = 1$, $\phi = -1$, the estimate of the point of maximum concentration is off by a factor of 2, but the value of the maximum concentration is approximately correct. This case was exhibited in Figure 3. It seems clear that the solution neglecting atmospheric diffusion predicts the value of the maximum concentration for relatively large ν much better than it predicts the location of the maximum. Typically, this yields curves of deposition which peak further downwind than the correct solution, and rather substantial differences in predicted deposition result--sometimes for considerable distances past the point of maximum concentration. This occurs even when (30) predicts the value of the maximum concentration correctly. For this reason is is best to use R^* or Figure 7 as the criteria rather than Λ^*.

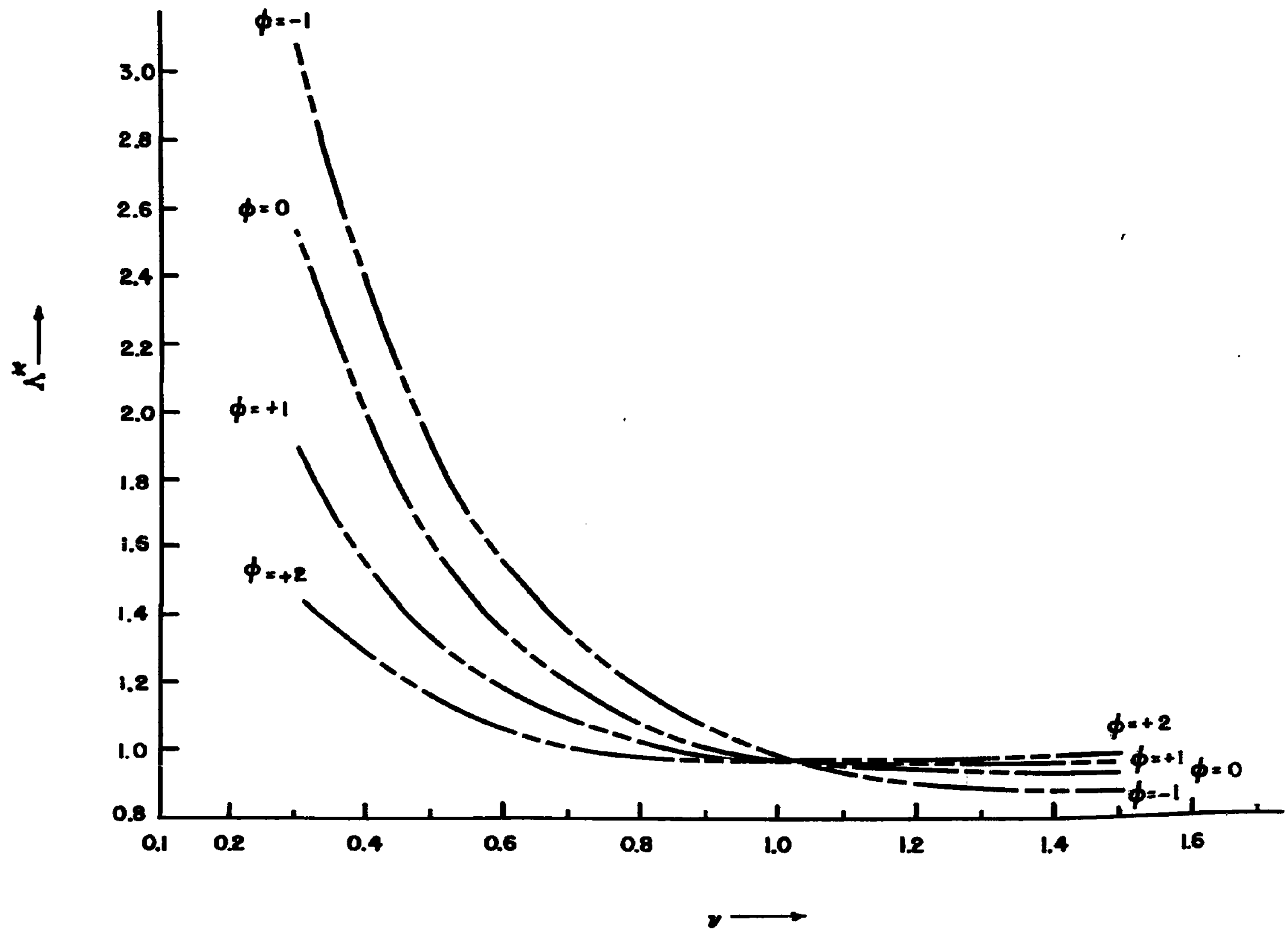

Figure 8. Λ^* as a function of ν for indicated values of ϕ .

4.2 When can variation of terminal velocity be neglected?

We are now prepared to treat the problem from another point of view. Given a group of particles with a mean terminal velocity, $\tilde{\Omega}$, under what circumstances is it permissible to neglect the variation of terminal velocity around the mean or median? For this case we may define the ratios $\bar{R}^*$, $\bar{\Lambda}^*$, $\tilde{R}^*$, and $\tilde{\Lambda}^*$ which are entirely similar to R^* and Λ^* defined previously, except that the numerators of the new ratios are given by (9) and (10) with

(41) $$p = \bar{\Omega}/\eta = e^{\phi + \frac{1}{2}\nu^2} ,$$

or

(42) $$p = \tilde{\Omega}/\eta = e^{\phi} .$$

Thus, using (41), in the definition of p, we have

(43) $$\bar{R}^* = \frac{1 + p^*}{1 + e^{\phi+\frac{1}{2}\nu^2}} \frac{f}{f^*} = \frac{e^{\frac{1}{2}\nu^2} R^*}{e^{-(\phi+\frac{1}{2}\nu^2)} + 1} .$$

If the median terminal velocity is employed in the definition of p, we have

(44) $$\tilde{R}^* = \frac{(1 + p^*)}{(1 + e^{\phi})} \frac{f}{f^*} = \frac{e^{\nu^2} R^*}{e^{-\phi} + 1} .$$

As in the case of R^* we can also consider the ratio of the predicted maximum deposit for particles with mean $\bar{\Omega}$ or mass median terminal velocity $\tilde{\Omega}$ to that predicted by (27). That is, we consider the ratio of (10) with p given by (41) or (42) to (27) and obtain

(45) $$\Lambda^* = \frac{\Gamma(p^*)}{\Gamma(e^{\phi+\frac{1}{2}\nu^2})} \frac{f^*}{f} \left(\frac{1+e^{\phi+\frac{1}{2}\nu^2}}{e}\right)^{1+e^{\phi+\frac{1}{2}\nu^2}} \left(\frac{e}{1 + p^*}\right)^{1+p^*}$$

$$\tilde{\Lambda}^{*} = \frac{\Gamma(p^{*})}{\Gamma(e^{\phi})}\,\frac{f^{*}}{f}\left(\frac{1+e^{\phi}}{e}\right)^{1+e^{\phi}}\left(\frac{e}{1+p^{*}}\right)^{1+p^{*}} \tag{46}$$

Inspection of (43) and (44) indicates that everything else being held constant, the percent error in the location of the maximum concentration is less when the mean rather than median terminal velocity is employed in the definition of p. Equation (43) is graphed in Figure 9 as a function of ν and ϕ. It may be seen that the error in the location of the maximum ground deposit is negligible when ν is small--as indeed it should be. The percent error becomes large as ν increases, but the results are still tolerable (less than 15% for $\nu < 0.55$ and $\phi < 0.5$) for surprisingly large values of ν.

Equation (45) is sketched in Figure 10. The value of the maximum concentration is apparently more sensitive to the spread of terminal velocity than is the point of maximum concentration. For fixed values of terminal velocity, and for small values of diffusion velocity (ϕ positive and large) it is apparent that the spread in terminal velocity cannot be neglected even when ν is as small as 0.3. However, when the diffusion velocity is large compared to the mean terminal velocity (ϕ negative), the dispersion of terminal velocity may be safely neglected. Thus when $\phi = -1$, the error in D_m and x_m is less than 15% for ν as large as 0.8.

5. Atmospheric Applications

5.1 Neutral conditions

It has been shown that the behavior of the ground deposition resulting from the release of particles whose mass distribution is log-

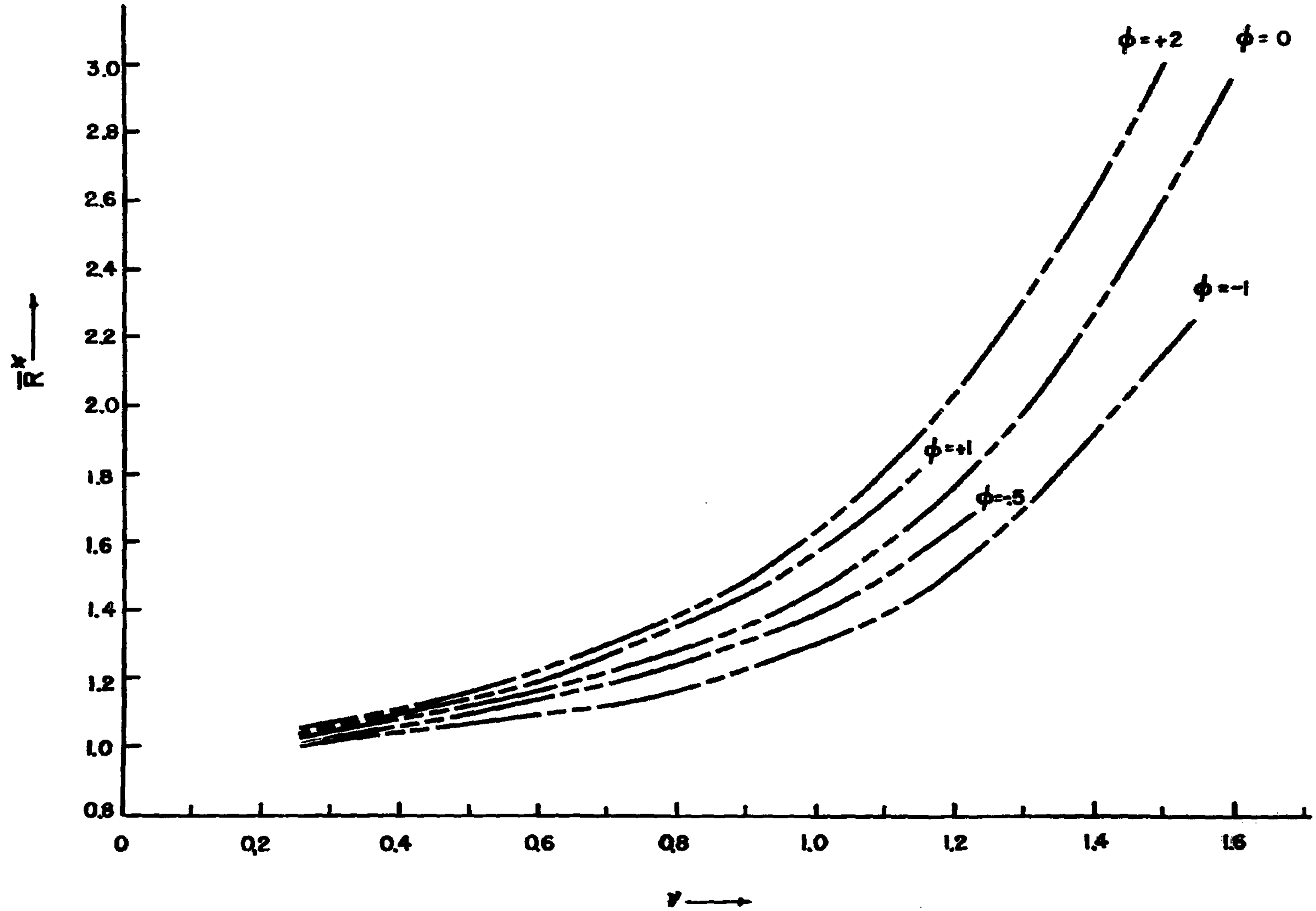

Figure 9. $\bar{R}^*$ as a function of ν for indicated values of ϕ .

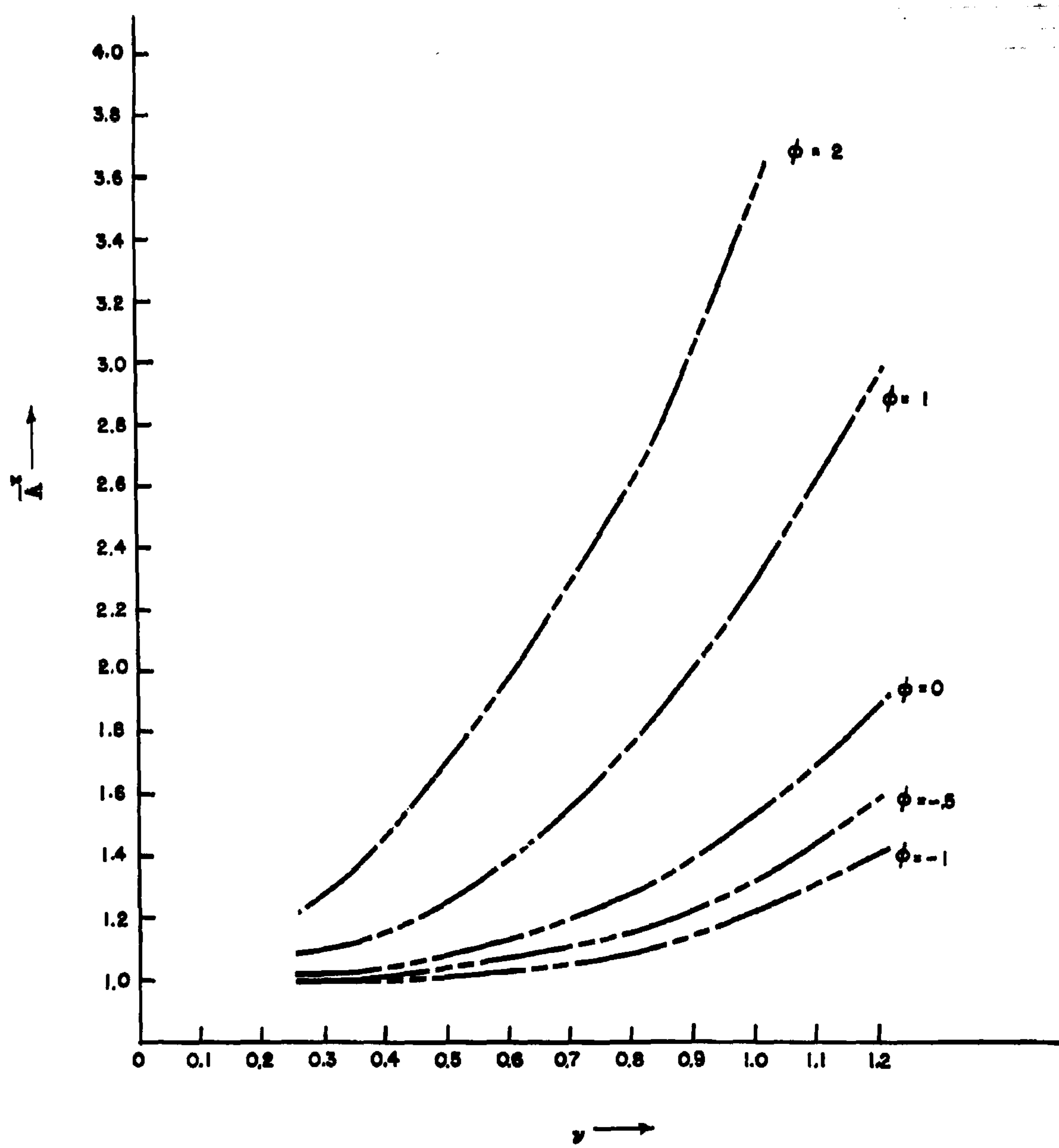

Figure 10. $\overline{\Lambda}^*$ as a function of ν for indicated values of ϕ .

normal with respect to terminal velocity depends upon the values of two parameters. The first parameter ϕ represents the logarithm of the ratio of the mass median terminal velocity to the virtual diffusion velocity, η. The second parameter, ν, depends on the dispersion of the terminal velocities around the logarithmic mean. The only meteorological parameter is $\eta = (1 + \alpha)ku^*$. Here $(1 + \alpha)$ is a measure of the wind shear which varies with atmospheric stability and ground roughness. The quantity u^* is, of course, proportional to the square root of the surface stress which, in turn, is a function of wind speed, ground roughness and stability.

The solutions presented thus far are valid for an adiabatic atmosphere. If $(1 + \alpha)$ is defined to yield the same average wind speed between ground level and height h <u>and</u> the same speed at release height h as that predicted by the more accurate logarithmic law, it follows that

$$(1 + \alpha) = \frac{\ln(h/z_o)}{\ln(h/z_o) - 1} \quad , \tag{47}$$

$$\eta = (1 + \alpha)ku^* = \frac{k^2}{\ln(h/z_o) - 1}\,\overline{u}(h) \quad , \tag{48}$$

and

$$\phi = \ln \frac{\widetilde{\Omega}[\ln(h/z_o) - 1]}{\overline{u}(h)k^2} \quad . \tag{49}$$

The parameter ϕ is therefore a function of (h/z_o), $\widetilde{\Omega}$ and $\overline{u}(h)$. The solid line plots in Figures 11 and 12 are curves of constant ϕ as functions of $\overline{u}(h)$ and $\widetilde{\Omega}$ for values of $h/z_o = 1.5 \times 10^3$ and 1.5×10^2 respectively. In terms of our previous work these values of h/z_o correspond to a release height of 15 m and a z_o of 1 cm and 10 cm respectively.

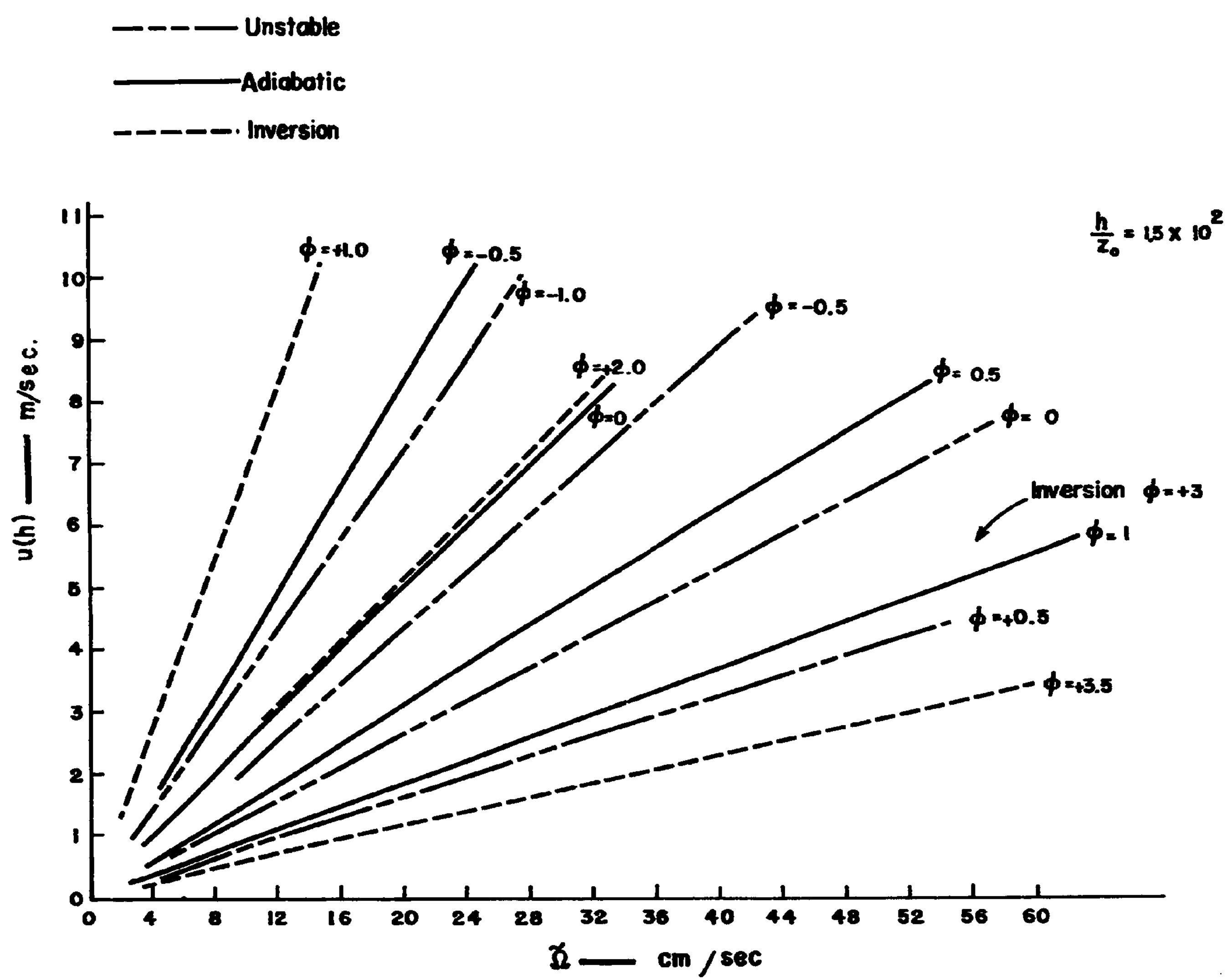

Figure 11. Lines of constant ϕ as a function of wind speed at height of release, mass median terminal velocity and atmospheric stability class for $h/z_o = 1.5 \times 10^2$.

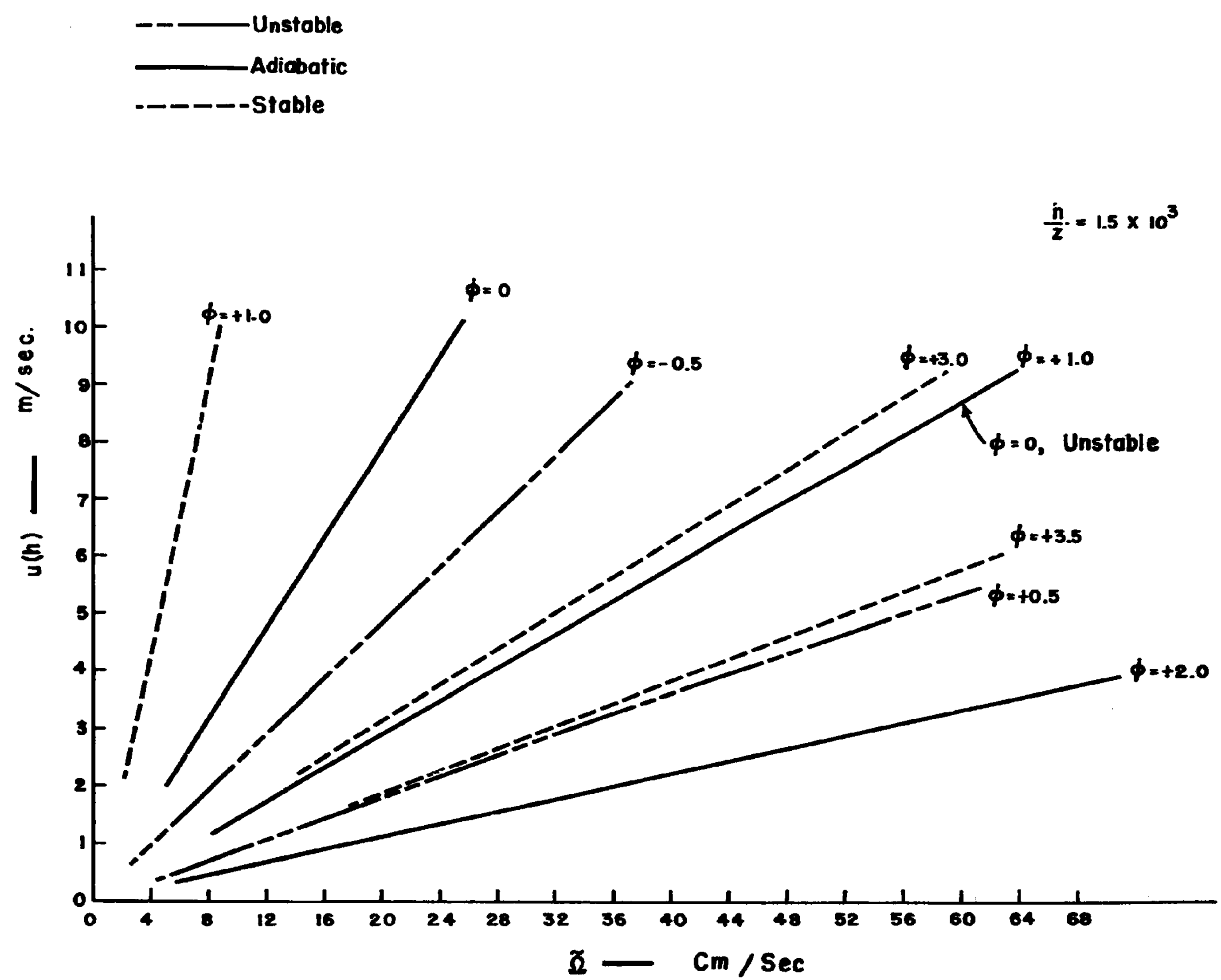

Figure 12. Lines of constant ϕ as a function of wind speed at height of release, mass median terminal velocity and atmospheric stability class for $h/z_o = 1.5 \times 10^3$

It may be seen that for fixed wind speed and mass median terminal velocity, the value of ϕ decreases as z_o increases. If $\tilde{\Omega}$ is 60 cm/sec and $\bar{u}(h)$ is 5 m/sec, $\phi \approx 1.2$ and 1.7 for $h/z_o = 1.5 \times 10^2$ and 1.5×10^3 respectively. Referring to Figures 7 and 8, it may be seen that for these values of ϕ, atmospheric diffusion may be neglected with fair approximation for values of $\nu > 0.6$ for the smooth terrain and 0.7 for the rougher terrain. Similarly, it may be seen from Figures 9 and 10, that for these values of $\bar{u}$ and $\tilde{\Omega}$, the effect of dispersion of terminal velocity on the diffusion pattern cannot be neglected except for values of ν less than 0.3. If $\tilde{\Omega}$ remains constant and the wind speed at release height increases to 10 m/sec, ϕ ranges between 0.4 and 0.9 and the range of validity of the two approximations, one neglecting atmospheric diffusion and the other neglecting terminal velocity dispersion, will change accordingly. It is interesting to note that the case $\tilde{\Omega} = 30$ cm/sec, $\bar{u} = 5$ m/sec corresponds to the same ϕ as $\tilde{\Omega} = 60$ cm/sec, $\bar{u} = 10$ m/sec.

This latter point illustrates the efficiency implicit in the non-dimensional plots of Figures 3-6. From the definition of (48), it is evident that a large number of possible $\tilde{\Omega}$, $\bar{u}(h)$ and h/z_o combinations can combine to yield the same value of ϕ. For any such combination yielding a fixed value of ϕ, say $\phi = 1.61$ and a ν value of 0.53, the appropriate curve of Figure 6 would be valid. The dimensional deposition may be recovered by multiplying the ordinate by Q/f. For an adiabatic atmosphere, using the definition of $(1 + \alpha)$ introduced at the beginning of this section, it follows that

$$f = \frac{H}{\eta} = \frac{h \cdot [\ln(h/z_o) - 1]^2}{k^2 \ln(h/z_o)} \cdot \mathrm{gm/(m\,sec)} \tag{50}$$

If, for example, the height of release were 15 m, and $z_o = 0.01$ m, then f would have the value of 511 m, and the dimensional deposition in gm/(m^{2}sec) for, say, Q=1000 gm/(m sec) would be recovered by multiplying the ordinate of the appropriate figure by Q/f = 1.96. The dimensional distance may be recovered by multiplying the x/f scale by 511 m. The subsidiary scales on Figure 6 indicate the values thus obtained. This case corresponds closely to the example presented in Progress Report 523.19 for $\tilde{\Omega} = e^{4.06} = 0.58$ m/sec and $\bar{u}(h) = 4.6$ m/sec.

5.2 Diabatic conditions

An approximate method of extending the monodispersed particle solution (7) to the diabatic atmosphere has been proposed by Godson (1958). For the diabatic atmosphere it is assumed, following Deacon (1949), that

$$\frac{K}{u_*} = kz_o \left(\frac{z}{z_o}\right)^{\beta} \quad , \tag{51}$$

and

$$\frac{\bar{u}}{u_*} = \frac{1}{k(1-\beta)} \left[\left(\frac{z}{z_o}\right)^{1-\beta} - 1 \right] \quad , \tag{52}$$

where β varies with stability in such a fashion that

$\beta < 1$, for stable conditions

$\beta = 1$, for adiabatic conditions

$\beta > 1$, for unstable conditions .

It has long ago been observed (Davidson and Barad, 1956; Panofsky, Blackadar and McVehil, 1960; and others) that the β formulation is qualitatively valid, but quantitatively incorrect. In view of the approxi-

mations that follow, the whole subsequent argument is semi-quantitative in nature, and probably does not warrant dealing with the somewhat more refined but nevertheless incorrect (for stable conditions) versions of the log-linear law, or the Ellison interpolation formula contained in the last referenced paper.

At any rate, (5) and (22) cannot be solved in terms of tabulated functions unless K is a linear function of height. Since z^β is linear only for $\beta = 1$ (adiabatic conditions), Godson suggested that an approximate way to deal with the problem is to find an ϵ such that

$$\int_0^h u^* k z_o \left(\frac{z}{z_o}\right)^\beta dz = \epsilon \int_0^h u^* kz \ .$$

It follows that when

$$\epsilon = \frac{2}{1+\beta}\left(\frac{z_o}{h}\right)^{1-\beta} ,$$

the mean value of the diffusion coefficient from zero to h under any stability conditions is reproduced by the linear diffusion coefficient

(53) $$K = ku^* \epsilon z \ .$$

Accordingly an approximate solution of (5) or (22) for diabatic conditions may be written exactly in the form (7) or (24) provided

(54) $$\eta_\beta = (1 + \alpha)k\epsilon u^* ,$$

$$p = \frac{\Omega}{\eta_\beta} , \qquad f = \frac{H}{\eta_\beta} .$$

From (52), it is apparent that

$$(1 + \alpha) = \frac{\overline{U}}{\overline{u}(h)} = \frac{1 - (z_o/h)^{1-\beta}}{\left[\frac{1}{2 - \beta} - \left(\frac{z_o}{h}\right)^{1-\beta}\right]}$$

and

$$u^* = \frac{(1 - \beta)\, k\overline{u}(h)}{(h/z_o)^{1-\beta} - 1} ,$$

where

(55) $$\overline{U} = \frac{1}{h}\int_0^h \overline{u}(h)dh$$

The parameter ϕ is now determined from the above as a function of h/z_o, β and $\overline{u}(h)$. The dashed lines on Figures 11 and 12 are lines of constant ϕ for $\beta = 1.10$. Experience indicates that this value of β corresponds to a case of marked instability and it is unlikely that the wind speed at release height will be greater than 5 m/sec for this value of β. It is apparent that the unstable atmosphere decreases the value of ϕ from its adiabatic value. For example, for $h/z_o = 1.5 \times 10^2$, $\widetilde{\Omega} = 60$ cm/sec, $\overline{u}(h) = 5$ m/sec, ϕ decreases from about 1.2 to about 0.5. Consequently, neglect of atmospheric diffusion is justified only for larger values of ν than was the case for the adiabatic atmosphere. Similarly, neglect of the effect of the variation of terminal velocity on the ground deposition pattern would be justified for slightly larger values of ν than was the case for the adiabatic atmosphere.

The dotted lines on Figures 11 and 12 represent constant values of ϕ for $\beta = 0.8$. This value of β corresponds to a moderately strong inversion. It may be seen that the values of ϕ are increased considerably over the adiabatic values. Consequently, during inversion conditipns,

the primary diffusing agent is the variation in terminal velocity, and neglect of atmospheric diffusion is valid for relatively small values of ν. On the other hand, a model which employs atmospheric diffusion and the average terminal velocity only, would be quite bad for even small values of ν.

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*These papers are not referred to in the text of this report since they appeared after the text was written. They contain material that is directly relevant to the contents of this paper.

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Statistical Laboratory, Research Division, College of Engineering, New York University, New York 53, N. Y. Field Assessment Problems "The Diffusion of Polydisperse Particulate Clouds: B. Davidson, L. Herbach.

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6. Probability

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II. Leon Herbach
III. Contract DA18-108-CML-6392 DA18-108-405-CML-526

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